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A coupled BEM and FEM for the interior transmission problem in acoustics

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ABSTRACT

The interior transmission problem (ITP) is a boundary value problem arising in inverse scattering theory, and it has important applications in qualitative methods. In this paper, we propose a coupled boundary element method (BEM) and a finite element method (FEM) for the ITP in two dimensions. The coupling procedure is realized by applying the direct boundary integral equation method to define the so-called Dirichlet-to-Neumann (DtN) mappings. We show the existence of the solution to the ITP for the anisotropic medium. Numerical results are provided to illustrate the accuracy of the coupling method.

1. Introduction

The interior transmission problem is a boundary value problem introduced in the inverse scattering theory for the study of the far field patterns for transmission problems [1,2]. It arises in the scattering of time-harmonic waves by an inhomogeneous medium of compact support. The interior transmission problem and the associated transmission eigenvalue problem have attracted much attention recently [3–5] because of their importance in the qualitative methods, such as the linear sampling method. In addition, the transmission eigenvalues can be determined from the far field pattern and be used to obtain estimates of the physical properties, such as the index of refraction, of the scattering object.

The ITP is a non-standard partial differential equation (PDE), and it is different from the classical acoustic transmission problem [6]. The problem is new and has not been covered by the classical PDE theory. In this paper, we focus on the ITP for the acoustic wave scattering by the anisotropic medium, and refer the readers to [7–9] and references therein for the interior transmission problem in the vector case, i.e., Maxwell's equations. Even though there are some papers discussing the theory of the ITP and the associated transmission eigenvalue problem, the study on numerical methods for these problems is quite limited. In this regard, some finite element methods to compute the transmission eigenvalues are reported in [10,11]. In this paper, we propose a coupled BEM and FEM for the ITP, and the coupling procedure is realized via the so-called Dirichlet-to-Neumann (DtN) mappings [12,13]. Concerning the coupling procedure of BEM and FEM, the first significant result addressing the theoretical justification was attained in [14]. A symmetric formulation for the coupling of BEM and FEM was proposed in [15,16]. The coupling methods also have been applied to time-harmonic scattering problems by many other authors [17–19].

The paper is organized as follows. In Section 2, we present the ITP for the anisotropic medium. In Section 3, we introduce the DtN mappings and corresponding non-local boundary value problems for the ITP. The existence of the solution is to

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be shown in Section 4 via the Fredholm alternative. In Section 5, we discuss the numerical implementation of the coupled method for the ITP, and present several numerical tests to show the accuracy of the method.

2. ITP for the anisotropic medium

Let $D \subset \mathbb{R}^2$ be an open bounded domain with a C^2 boundary $\Gamma = \partial D$. Let A be a symmetric matrix valued function in $\bar{D}$ such that $\xi \cdot \mathrm{Im}(A)\xi \le 0$ and $\xi \cdot \mathrm{Re}(A)\xi \ge \gamma|\xi|^2$ for all $\xi \in \mathbb{R}^2$ and $x = (x_1, x_2) \in \bar{D}$ with $\gamma > 0$. For a function $u \in C^1(\bar{D})$ the conormal derivative is defined by

$$\frac{\partial u}{\partial \nu_A}(x) := \nu(x) \cdot A(x)\nabla u(x), \quad x \in \Gamma,$$

where ν is the unit outward normal to Γ. The interior transmission problem can be stated as follows. For given functions $f \in H^{1/2}(\Gamma)$ and $g \in H^{-1/2}(\Gamma)$, we need to find two functions $v \in H^1(D)$ and $w \in H^1(D)$ satisfying

$$\nabla \cdot A\nabla w + k^2 n(x) w = 0 \quad \text{in } D, \tag{2.1a}$$

$$\Delta v + k^2 v = 0 \quad \text{in } D, \tag{2.1b}$$

$$w - v = f \quad \text{on } \Gamma, \tag{2.1c}$$

$$\frac{\partial w}{\partial \nu_A} - \frac{\partial v}{\partial \nu} = g \quad \text{on } \Gamma. \tag{2.1d}$$

Here, we further assume [20] that the ellipticity constant $\gamma > 1$ for $x \in D$. The interior transmission problem serves as an important tool to study the kernel of the far field operator [3].

Definition 2.1. If $k > 0$ is such that the homogeneous interior transmission problem has a nontrivial solution, then k is called a transmission eigenvalue.

3. Non-local boundary value problems

In this section, we will reduce the interior transmission problem (2.1a)–(2.1d) to two non-local boundary value problems in D in terms of two different forms of the DtN mapping.

3.1. Dirichlet-to-Neumann mappings

To construct the DtN mappings on Γ, we first consider the following interior Dirichlet problem. Given the function $\phi \in H^{1/2}(\Gamma)$, we find $v \in H^1(D)$ satisfying

$$\Delta v + k^2 v = 0 \quad \text{in } D, \tag{3.2a}$$

$$v = \phi, \quad \text{on } \Gamma. \tag{3.2b}$$

The solution v of the boundary value problem (3.2) can be represented by Green's representation formula in terms of the fundamental solution

$$E(x, y) = \frac{i}{4} H_0^{(1)}(k|x - y|), \tag{3.3}$$

for the two-dimensional Helmholtz equation. Here, $H_0^{(1)}$ is the Hankel function of the first kind of order zero. This representation assumes the form

$$v(x) := \int_\Gamma E(x, y)\sigma(y)\, \mathrm{d}s_y - \int_{\partial D} \frac{\partial E(x, y)}{\partial \nu_y}\mu(y)\, \mathrm{d}s_y, \quad \forall x \in D, \tag{3.4}$$

where

$$\mu = v^-, \quad \text{and} \quad \sigma = \frac{\partial v}{\partial \nu_y}^-$$

denote the Cauchy data on Γ for the solution v. Here, and in the sequel, we write w^- for the boundary limit of any function or distribution w defined in D. Letting x in (3.4) approach the boundary Γ, and employing the jump conditions, we obtain the boundary integral equation (BIE)

$$V\sigma(x) = \left(\frac{1}{2} I + K\right)\mu(x), \quad \forall x \in \Gamma. \tag{3.5}$$

Here, I stands for the identity operator, and V and K are basic simple- and double-layer boundary integral operators defined by

$$V\sigma(x) = \int_\Gamma E(x,y)\sigma(y)\mathrm{d}s_y, \quad \forall x \in \Gamma, \tag{3.6}$$

$$K\mu(x) = \int_\Gamma \frac{\partial E(x,y)}{\partial \nu_y}\mu(y)\mathrm{d}s_y, \quad \forall x \in \Gamma, \tag{3.7}$$

respectively. Prior to defining the first DtN mapping (also known as the *Steklov–Poincaré* operator) in terms of boundary integral operators, we state some of the mapping properties for the boundary integral operators V and K on the *Sobolev* spaces $H^s(\Gamma)$ for $s = -1/2, 1/2$ [21]. For sufficiently smooth boundary Γ, we have

1. The simple-layer boundary integral operator V is an isomorphism from $H^{-1/2}(\Gamma)$ to $H^{1/2}(\Gamma)$, if and only if k^2 is not an eigenvalue for the interior Dirichlet problem for the negative Laplacian $-\Delta$ in D.

2. The double-layer boundary integral operator K is a continuous mapping from $H^{1/2}(\Gamma)$ to $H^{3/2}(\Gamma)$. In particular, due to the Rellich compactness theorem, the operator K is compact from $H^{1/2}(\Gamma)$ to $H^{1/2}(\Gamma)$.

With the properties of boundary integral operators V and K, we now define the first DtN mapping $T : H^{1/2}(\Gamma) \to H^{-1/2}(\Gamma)$ as

$$T\varphi := V^{-1}\left(\frac{1}{2}I + K\right)\varphi(x), \quad \forall \varphi \in H^{1/2}(\Gamma). \tag{3.8}$$

It is worth mentioning that the reduction of the boundary value problem (3.2) to an equivalent boundary integral equation is not generally a unique process [12]. In the following, we define another DtN mapping by employing the symmetric coupling procedure [15,16].

By computing the normal derivative for both sides of the representation formula (3.4) and taking the limits as $x \to \Gamma$, we arrive at the second boundary integral equation

$$\sigma(x) = \left(\frac{1}{2}I + K'\right)\sigma(x) + W\mu(x), \quad \forall x \in \Gamma, \tag{3.9}$$

where K' is the transpose of K in (3.7) and W is the hypersingular boundary integral operator, and they are defined as

$$K'\sigma(x) = \int_\Gamma \frac{\partial E(x,y)}{\partial \nu_x}\sigma(y)\mathrm{d}s_y, \quad \forall x \in \Gamma, \tag{3.10}$$

$$W\mu(x) = -\frac{\partial}{\partial \nu_x}\int_\Gamma \frac{\partial E(x,y)}{\partial \nu_y}\mu(y)\mathrm{d}s_y, \quad \forall x \in \Gamma. \tag{3.11}$$

Similarly, the properties of K' and W are presented as follows [21,22].

1. The boundary integral operator K' is a continuous mapping from $H^{-1/2}(\Gamma)$ to $H^{1/2}(\Gamma)$.

2. The hypersingular boundary integral operator W is a continuous mapping from $H^{1/2}(\Gamma)$ to $H^{-1/2}(\Gamma)$.

In terms of boundary integral operators K', W, V and K, we may now define an alternative DtN mapping $T : H^{1/2}(\Gamma) \to H^{-1/2}(\Gamma)$ as

$$T\varphi := \left(\frac{1}{2}I + K'\right)V^{-1}\left(\frac{1}{2}I + K\right)\varphi + W\varphi, \quad \forall \varphi \in H^{1/2}(\Gamma), \tag{3.12}$$

if V is invertible.

Theorem 3.1. *The DtN mapping T in (3.8) or (3.12) is a bounded linear operator from $H^{1/2}(\Gamma)$ to $H^{-1/2}(\Gamma)$, if and only if k^2 is not an eigenvalue of the interior Dirichlet problem for the negative Laplacian $-\Delta$ in D.*

3.2. Non-local boundary value problems for w

With these DtN mappings, applying the transmission conditions (2.1c) and (2.1d), and replacing ϕ in (3.2b) by $w^- - f$ to eliminate v, we can reduce the ITP to non-local boundary value problems for w consisting of (2.1a), namely,

$$\nabla \cdot A\nabla w + k^2 n(x) w = 0 \quad \text{in } D, \tag{3.13}$$

and the non-local boundary condition

$$\frac{\partial w}{\partial \nu_A} = \frac{\partial v}{\partial \nu} + g = Tw + (g - Tf). \tag{3.14}$$

Therefore, the existence of the solution for the ITP amounts to the existence of the solution for the non-local boundary value problem (3.13) and (3.14).

4. Existence of solution

For the proof of existence, let us consider the following modified ITP (MITP)

$$\nabla \cdot A\nabla w - mw = 0 \quad \text{in } D, \tag{4.15a}$$

$$\Delta v - v = 0 \quad \text{in } D, \tag{4.15b}$$

$$w - v = f \quad \text{on } \Gamma, \tag{4.15c}$$

$$\frac{\partial w}{\partial \nu_A} - \frac{\partial v}{\partial \nu} = g \quad \text{on } \Gamma. \tag{4.15d}$$

Here, we have assumed that $m(x) \geq \gamma > 1$. We first reduce the MITP to a non-local problem for w by following the procedure used for the original ITP in the previous section. Then, we show that the solution of the non-local problem for the MITP exists. Finally, we show that the non-local boundary value problem (3.13) and (3.14) is a compact perturbation of the corresponding non-local boundary value problem for the MITP. The existence for the solution of the ITP thus follows by applying the Fredholm alternative: uniqueness implies existence.

We define a DtN mapping T_0 as

$$T_0\phi := V_0^{-1}\left(\frac{1}{2}I + K_0\right)\phi, \quad \text{for } \phi \in H^{1/2}(\Gamma),$$

where V_0 and K_0 correspond to the simple- and double-layer boundary integral operators in terms of the fundamental solution $E_0(x, y) = \frac{1}{2\pi}\Phi_0(|x - y|)$ of

$$-\Delta u + u = 0.$$

Here, Φ_0 is the modified Bessel function of the second kind of order zero. Then the MITP is reduced to a non-local boundary value problem consisting of (4.15a) and the non-local boundary condition

$$\frac{\partial w}{\partial \nu_A} = T_0 w + (g - T_0 f) \quad \text{on } \Gamma. \tag{4.16}$$

Now we consider the weak formulation of (4.16) and (4.15a). The sesquilinear form is defined as

$$a(w, \hat{w}) := \int_D A\nabla w \cdot \nabla\bar{\hat{w}}\, \mathrm{d}x + \int_D m w \bar{\hat{w}}\, \mathrm{d}x - \langle T_0 w, \bar{\hat{w}}\rangle_\Gamma.$$

And clearly,

$$\mathrm{Re}\{a(w, w)\} \geq \gamma\|\nabla w\|^2_{L^2(D)} + \mathrm{Re}\left\{\int_D m w\bar{w}\, \mathrm{d}x\right\} - \mathrm{Re}\{\langle T_0 w, \bar{w}\rangle_\Gamma\}, \quad \forall w \in H^1(D). \tag{4.17}$$

We see by construction that

$$\int_D (\nabla w \cdot \nabla\bar{w} + w\bar{w})\mathrm{d}x = \langle T_0 w, \bar{w}\rangle_\Gamma,$$

i.e.,

$$\langle T_0 w, \bar{w}\rangle_\Gamma = \|w\|^2_{H^1(D)}$$

from Green's identity. Consequently, (4.17) leads to

$$\begin{aligned}\mathrm{Re}\{a(w, w)\} &\geq \gamma(\|\nabla w\|^2_{L^2(D)} + \|w\|^2_{L^2(D)}) - \|w\|^2_{H^1(D)} \\ &= (\gamma - 1)\|w\|^2_{H^1(D)}\end{aligned} \tag{4.18}$$

because $m(x) \geq \gamma$. This implies the existence of solution of (4.16) and (4.15a) as $\gamma > 1$. Thus we have proved the following theorem.

Theorem 4.1. *Assume that $\xi \cdot \mathrm{Re}(A)\xi \geq \gamma|\xi|^2$ for all $\xi \in \mathbb{R}^2$ and $x \in \bar{D}$ and $m(x) \geq \gamma > 1$. Then, there exists a unique solution to the modified interior transmission problem (4.15).*

Now we return to the non-local boundary value problem for the original ITP. The weak form for the non-local problem (3.13) and (3.14) reads as follows. For given functions $f \in H^{1/2}(\Gamma)$ and $g \in H^{-1/2}(\Gamma)$, looking for $w \in H^1(D)$ such that

$$a(w, \hat{w}) = \ell(\hat{w}) \tag{4.19}$$

for all $\hat{w} \in H^1(D)$. Here, $a(w,\hat{w}) = \int_D (A\nabla w \cdot \nabla \bar{\hat{w}} - k^2 n w \bar{\hat{w}}) \mathrm{d}x - \langle Tw, \bar{\hat{w}}\rangle_\Gamma$, and $\ell(\hat{w}) = \langle g - Tf, \bar{\hat{w}}\rangle_\Gamma$. Moreover, the sesquilinear form $a(w,w)$ can be rewritten in the form

$$a(w,w) = \int_D A\nabla w \cdot \nabla \bar{w}\, \mathrm{d}x + \int_D m w \bar{w}\, \mathrm{d}x - \langle T_0 w, \bar{w}\rangle_\Gamma - \left\{ \int_D (k^2 n + m) w \bar{w}\, \mathrm{d}x + \langle (T - T_0) w, \bar{w}\rangle_\Gamma \right\}.$$

Therefore, we have

$$\mathrm{Re}\{a(w,w)\} \geq (\gamma - 1)\|w\|^2_{H^1(D)} - \mathrm{Re}\{(\mathcal{C}w, w)_{H^1(D)}\}$$

because of (4.17) and (4.18). Here, the operator $\mathcal{C} : H^1(D) \to H^1(D)$ is a compact operator defined by

$$(\mathcal{C}w, \hat{w})_{H^1(D)} := \int_D (k^2 n + m) w \bar{\hat{w}}\, \mathrm{d}x + \langle (T - T_0) w, \bar{\hat{w}}\rangle_\Gamma.$$

To see that $\mathcal{C} : H^1(D) \to H^1(D)$ is compact, we decompose $\mathcal{C}$ as $\mathcal{C} = \mathcal{C}_1 + \mathcal{C}_2$ with

$$(\mathcal{C}_1 w, \hat{w})_{H^1(D)} := ((k^2 n + m) w, \hat{w})_{L^2(D)}$$

and

$$(\mathcal{C}_2 w, \hat{w})_{H^1(D)} := \langle (T - T_0) w, \hat{w}\rangle_\Gamma,$$

and then show the compactness of the operators $\mathcal{C}_1$ and $\mathcal{C}_2$, respectively.

First of all, we know that

$$(\mathcal{C}_1 w, \hat{w})_{H^1(D)} = (w, \mathcal{C}_1^* \hat{w})_{H^1(D)},$$

and

$$|(w, \mathcal{C}_1^* \hat{w})_{H^1(D)}| \leq c_0 \|w\|_{H^1(D)} \|\hat{w}\|_{L^2(D)}. \tag{4.20}$$

The latter further implies that

$$\|\mathcal{C}_1^* \hat{w}\|_{H^1(D)} \leq c_0 \|\hat{w}\|_{L^2(D)},$$

where $\mathcal{C}_1^*$ is the adjoint operator of $\mathcal{C}_1$, and $c_0 = \max_{x\in\bar{D}} |(k^2 n + m)(x)|$. It follows that $\mathcal{C}_1^*$ maps $L^2(D)$ into $H^1(D)$ continuously. Since Lipschitz domains enjoy the uniform cone property, Rellich's Lemma implies the compactness of

$$\mathcal{C}_1^* : H^1(D) \to H^1(D),$$

and hence $\mathcal{C}_1 : H^1(D) \to H^1(D)$ is compact as well.

We now show the compactness of the operator $\mathcal{C}_2 : H^1(D) \to H^1(D)$. From the series development of the fundamental solutions, we see that

$$E(x,y) - E_0(x,y) = \text{const.} + O(kr^2 \log(kr^2)),$$

where $r = |x - y|$. Then the difference of the corresponding DtN mapping, $T - T_0$, is given explicitly in the form

$$(T - T_0)\varphi := V_0^{-1}\{(K - K_0) - (V - V_0)T\}\varphi$$

which is smoother than T. Hence the operator $T - T_0 : H^{1/2}(\Gamma) \to H^{-1/2}(\Gamma)$ is compact. Meanwhile, the operator $\mathcal{C}_2$ is a composite operator of the following continuous operators

$$H^1(D) \xrightarrow{\gamma_0} H^{1/2}(\Gamma) \xrightarrow{T-T_0} H^{-1/2+\epsilon}(\Gamma) \xrightarrow{e} H^{-1/2}(\Gamma) \xrightarrow{j} H^{1/2}(\Gamma) \xrightarrow{\gamma_0^{-1}} H^1(D)$$

for some small $\epsilon > 0$. Here, γ_0 and γ_0^{-1} denote the trace and right inverse of the trace operators, $\xrightarrow{e}$ the compact embedding and j the Riesz mapping. Consequently, $\mathcal{C}_2 : H^1(D) \to H^1(D)$ is compact.

Finally, we point out that if T in (3.8) is replaced by T in (3.12) and T_0 by the corresponding T_0 according to (3.12), the proof of the compactness of C_2 follows by the same argument.

Now, we see that $a(w,\hat{w})$ satisfies a Gårding's inequality in the form

$$\mathrm{Re}\{a(w,w) + (\mathcal{C}w, w)_{H^1(D)}\} \geq c\|w\|^2_{H^1(D)} \quad \forall w \in H^1(D), \tag{4.21}$$

where $c = \gamma - 1 > 0$ is a constant independent of $w \in H^1(D)$. As a consequence, the existence follows from the Fredholm alternative: uniqueness implies existence. Thus we have established the following theorem.

Theorem 4.2. *Assume that $\xi \cdot \mathrm{Re}(A)\xi \geq \gamma|\xi|^2$ for all $\xi \in \mathbb{R}^2$ and $x \in \bar{D}$ for some $\gamma > 1$. Let $f \in H^{1/2}(\Gamma)$ and $g \in H^{-1/2}(\Gamma)$. Then, there exists a unique solution to the interior transmission problem (2.1), if k is not a transmission eigenvalue.*

5. Numerical scheme

In this section, we employ the finite element method for the numerical solution of the variational equation (4.19).

5.1. Galerkin formulation

Let S_h be the standard finite element space. Now we consider the Galerkin formulation of (4.19). Given $f \in H^{1/2}(\Gamma)$ and $g \in H^{-1/2}(\Gamma)$, find $w_h \in S_h \subset H^1(D)$ satisfying

$$a(w_h, v_h) = \ell(v_h), \quad \forall\, v_h \in S_h. \tag{5.22}$$

We can show [21] that the discrete sesquilinear form $a(w_h, v_h)$ satisfies the BBL-condition as implication of the following:
Gårding's inequality + Uniqueness + Approximation property of $S_h \Rightarrow$ *BBL-condition.*

Theorem 5.1. *If the sesquilinear form $a(w, v)$ in (4.19) satisfies the following conditions,*

1. $Re\{a(w, w) + (\mathcal{C}w, w)_{H^1(D)}\} \geq \alpha \|w\|^2_{H^1(D)}, \ \forall w \in H^1(D)$;
2. $\{w \in H^1(D) | a(w, v) = 0, \ \forall v \in H^1(D)\} = \{0\}$;
3. *Finite element space $S_h \subset H^1(D)$ satisfies the standard approximation property.*

Then, there exists a constant $h_0 > 0$ such that $a(w, v)$ for $0 < h \leq h_0$ satisfies the BBL-condition in the form

$$\sup_{0 \neq v_h \in S_h} \frac{|a(w_h, v_h)|}{\|v_h\|_{H^1(D)}} \geq \beta \|w_h\|_{H^1(D)}, \quad \forall w_h \in S_h. \tag{5.23}$$

Here, $\mathcal{C}$ is a compact operator from $H^1(D)$ to $H^1(D)$, $\alpha > 0$ is a constant, and $\beta > 0$ is the inf–sup constant independent of h.

By the BBL-condition (5.23), the unique solvability of the Galerkin equation (5.22), and the convergence of its Galerkin solution as $h \to 0^+$ can be obtained accordingly (see Theorem 1 in [14]). The numerical implementation for (5.22) can be found in many finite element books, and next, we introduce only how the boundary element method is applied.

5.2. Computation of $\langle Tw, \hat{w}\rangle_\Gamma$

To find the finite element solution of (5.22), we must be able to numerically evaluate the sesquilinear form $\langle Tw, \hat{w}\rangle_\Gamma$. In the discrete formulation, this amounts to computing the integrals

$$-\int_\Gamma (T\varphi_j)\varphi_i \mathrm{d}s, \tag{5.24}$$

where $\varphi_i, \ i = 1, 2, \dots, N$, are basis functions of the finite element space S_h. Here, N is the number of degrees of freedom. In this work, we only compute (5.24) via the definition of T in (3.8) for which we need to solve the N boundary integral equations

$$\frac{\partial \varphi_j}{\partial \nu} = T\varphi_j = V^{-1}\left(\frac{1}{2}I + K\right)\varphi_j, \quad j = 1, 2, \dots, N, \tag{5.25}$$

and then compute the integral (5.24) using appropriate quadrature rules. In particular, we employ the Galerkin boundary element method for the numerical solution of (5.25). The computational task is formidable in the general case since one has to solve N boundary integral equations. In our simulations, the finite element space consists of piecewise linear functions in P^1, and most of them vanish on the boundary Γ correspondingly eliminating the complexity of the above procedure. Here, P^1 is the space of polynomials with the total degree at most one.

To compute the Galerkin solution of (5.25), we choose the piecewise linear and constant basis functions to form the finite-dimensional subspaces of $H^{1/2}(\Gamma)$ and $H^{-1/2}(\Gamma)$, respectively, and the piecewise constant basis functions as test functions. Suppose that the Galerkin equation (5.22) is computed on the mesh M_1. Let the mesh M_2 be the refinement of M_1 (we use the PDE tool box of MATLAB to produce the mesh and make the refinement). And the Galerkin equation of (5.25) is computed on the boundary elements of M_2. We then take the arithmetic average of $T\varphi_j$ on the boundary elements of M_2 to give the value of $T\varphi_j$ on the boundary elements of M_1. Referring to the left column in Fig. 1 (or Fig. 2), for instance, if M_1 denotes the top mesh, then M_2 is the middle mesh. We employ the one-point Gaussian quadrature for the evaluation of integrals during the discretization of (5.25).

5.3. Numerical examples

In this section, we compute three numerical examples to illustrate the practicability of the coupling procedure. The first example provides a model for which the analytic solution is available for validating the accuracy of the method. Let D be the

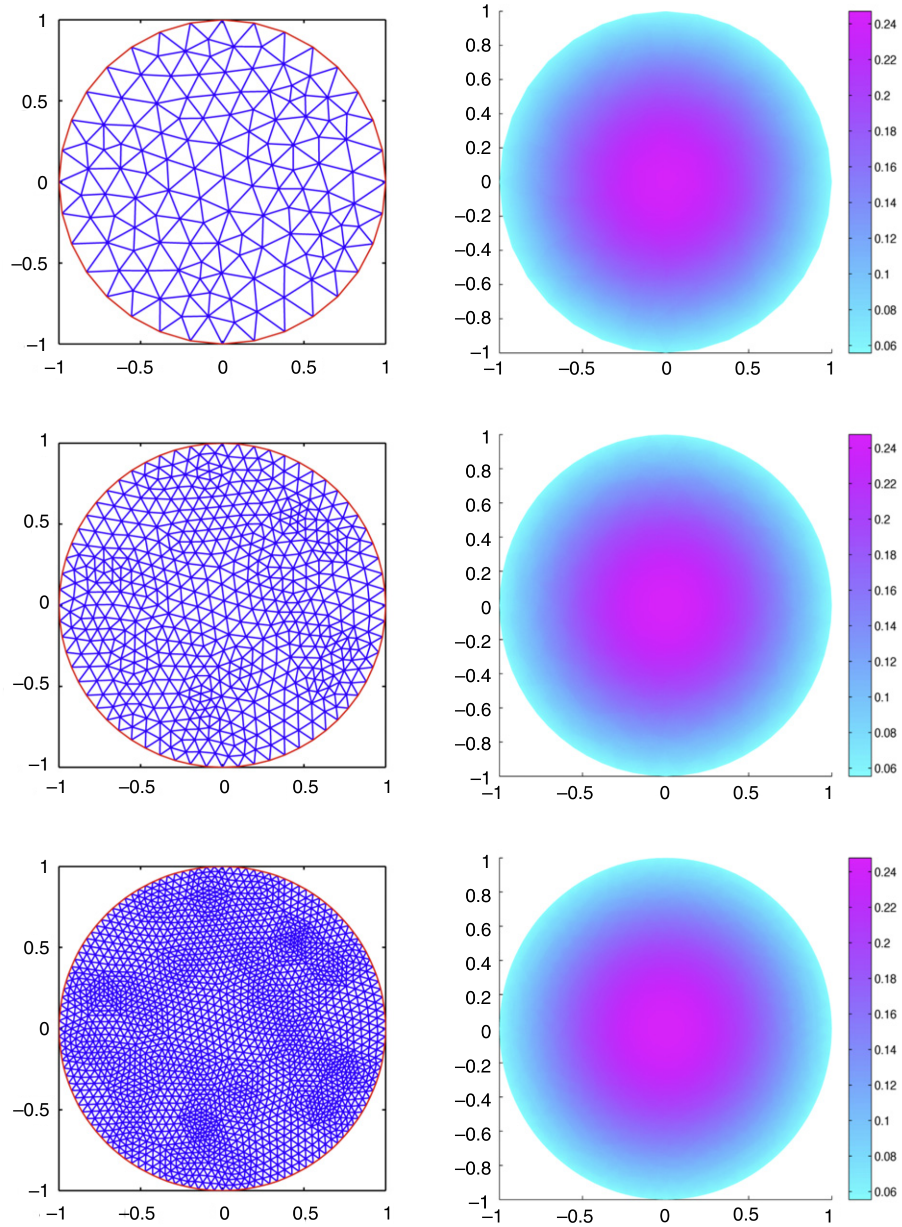

Fig. 1. P^1 solutions of the second example. From the top to the bottom: coarse mesh; fine mesh; finest mesh.

unit disk and $k = 1$. Let $A = \mathrm{diag}(2, 2)$ and $n(x) = 8$ on D. The Bessel functions $w = J_0(2r)$ and $v = J_0(r)$ are the solutions of (2.1a) and (2.1b) on D, respectively. Therefore $w = J_0(2r)$ and $v = J_0(r)$ solve the interior transmission problem (2.1) with $f = J_0(2) - J_0(1)$, $g = -4J_1(2) + J_1(1)$.

We compute the relative discrete L^2-error on the unit disk via

$$\text{relative error} = \frac{\sqrt{\sum_{i=1}^{NP}(w_i - w_i^{\mathrm{ex}})^2}}{\sqrt{\sum_{i=1}^{NP}(w_i^{\mathrm{ex}})^2}}, \tag{5.26}$$

where NP is the number of points on the unit disk, w_i and w_i^{ex} are the finite element approximations and exact solutions, respectively. In Table 1, we present the relative errors and the order of accuracy for the P^1 approximations, and one can see that the expected second order accuracy has been obtained.

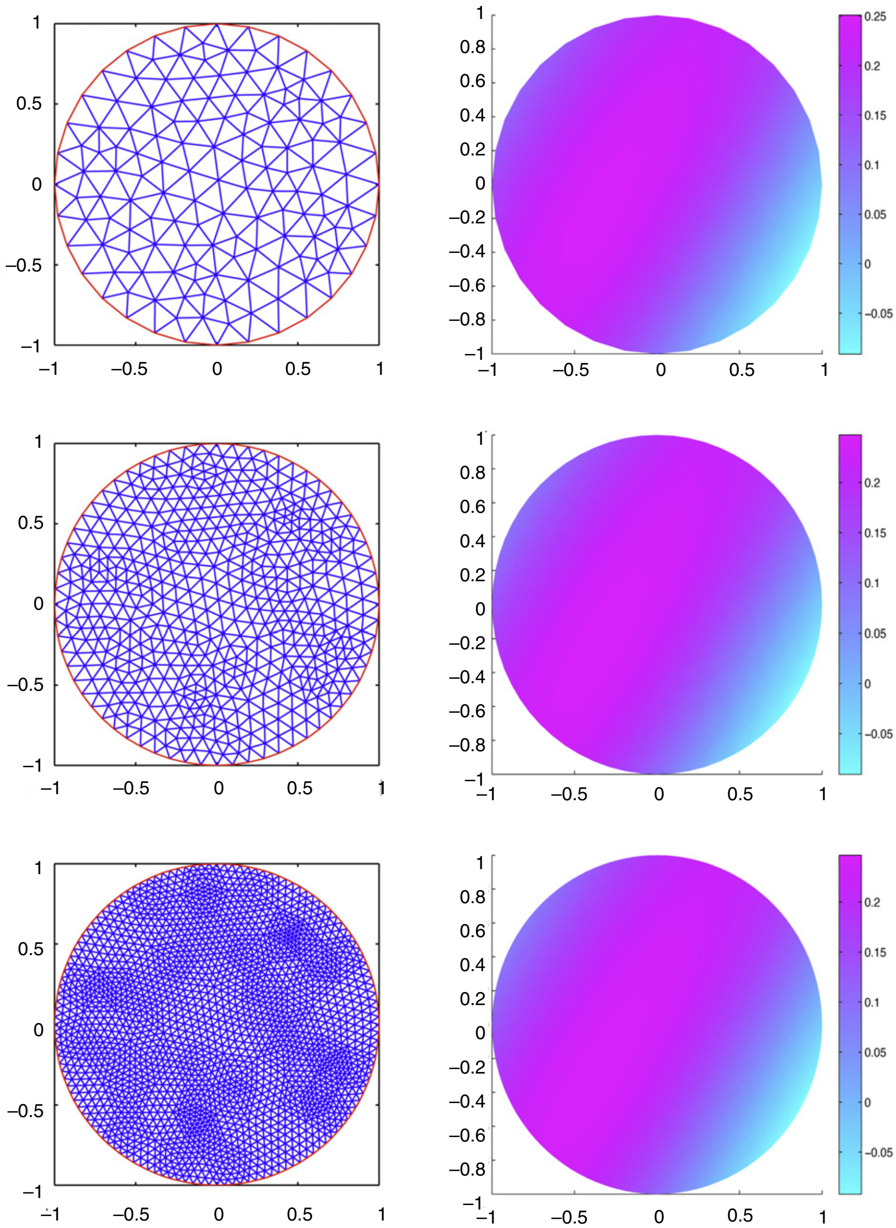

Fig. 2. P^1 solutions of the third example. From the top to the bottom: coarse mesh; fine mesh; finest mesh.

The second example has been considered in [1] to determine the refractive index n. Let D be the unit disk, $k = 1$, $A = I$, $n = 4$, and $\gamma = 1$. And functions f and g are given by

$$f = v - w = \frac{1}{r}\mathrm{e}^{-\mathrm{i}kr} \quad \text{on } \Gamma,$$
$$g = \frac{\partial v}{\partial \nu} - \frac{\partial w}{\partial \nu} = \frac{\partial}{\partial \nu}\left(\frac{1}{r}\mathrm{e}^{-\mathrm{i}kr}\right) \quad \text{on } \Gamma.$$

Thus

$$f = \mathrm{e}^{-\mathrm{i}}, \qquad g = -\mathrm{e}^{-\mathrm{i}} - \mathrm{i}\mathrm{e}^{-\mathrm{i}} \quad \text{on } \Gamma.$$

Note that for this case there are no exact solutions. We perform numerical computations on three different meshes (left column in Fig. 1), and present corresponding numerical results on the right column in Fig. 1. It can be observed that the numerical solution is stable and convergent as the mesh is refined.

Table 1
Relative L^2 error and order of accuracy for P^1 approximations of the first example. h is the mesh size in finite element discretization.

h	Relative error	Order
2.34E−1	1.03E−2	
1.21E−1	2.64E−3	2.06
6.13E−2	6.00E−4	2.18
3.09E−2	1.39E−4	2.14

We now investigate the third example with anisotropic properties. Let D be the unit disk and $k = 1$. We choose

$$A = \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix}, \quad \text{and} \quad n(x) = 4 + x_1 - x_2.$$

f and g are given by

$$f = \mathrm{e}^{-\mathrm{i}}, \qquad g = -\mathrm{e}^{-\mathrm{i}} - \mathrm{i}\mathrm{e}^{-\mathrm{i}} \quad \text{on } \Gamma.$$

We again perform the numerical computations on three different meshes (left column in Fig. 2), and present corresponding numerical results on the right column in Fig. 2. Clearly, the stability and convergence of the numerical solutions can be observed as the mesh is refined.

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